No-contact results for fluid-plate interaction

Free Boundary Problems 2026

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Joint work with

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on sufficient conditions which ensure that the elastic plate forming the upper boundary of a viscous fluid domain remains separated from the rigid bottom of a **three-dimensional** periodic channel.

based on a true story:

- [BKLM-1] A global existence result on weak solutions for the 3D Navier–Stokes–plate system with no contact. **J. Nonlinear Sci.** 36, 60 (2026).
- [BKLM-2] A no-contact result for a plate–fluid interaction system in dimension three. To appear in **SIAM J. Math. Anal.** (2026).

Introduction

Motivation

- long-term objective: rigorous derivation of the 2D elastohydrodynamic lubrication (EHL) model from the 3D Navier–Stokes–plate system in a thin fluid layer:

$$\text{3D fluid–plate interaction} \quad \xrightarrow{\varepsilon = H/L \to 0} \quad \text{2D EHL}$$

- 1D EHL: quantitative energy and distance estimates for fluid–beam systems prevent contact in finite time [Grandmont and Hillairet, ARMA 2016; B. and Muha, Nonlinearity 2022]
- challenges: the analogous argument does not directly extend from 2D→1D to 3D→2D:
  - the natural energy gives only H^2 control of plate deformation;
  - the 1D interpolation mechanism controlling $\|\eta^{-1}\|_{L^\infty}$ fails in 2D

no-contact results are a necessary first step toward a rigorous EHL

Introduction

Fluid–plate interaction model

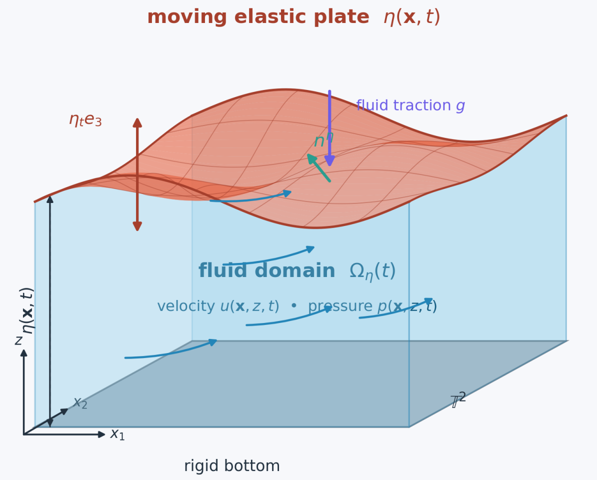


Figure: GPT-5.6

- $\eta \colon \mathbb{T}^2 \times [0, +\infty) \to \mathbb{R}$ - height of the moving boundary and $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$
- fluid domain:

$$\Omega_\eta(t) = \{(x, z) : x = (x_1, x_2) \in \mathbb{T}^2, z \in (0, \eta(x, t))\} \subseteq \mathbb{R}^3.$$

- fluid motion:

$$\begin{aligned} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 \qquad & \text{on } \Omega_\eta(t) \times (0, T), \\ \operatorname{div} u = 0 \qquad & \text{on } \Omega_\eta(t) \times (0, T), \end{aligned}$$

  where $u \colon \Omega_\eta(t) \times (0, T) \to \mathbb{R}^3$ and $p \colon \Omega_\eta(t) \times (0, T) \to \mathbb{R}$ denote the velocity and pressure.
- plate motion:

  $$\eta_{tt} - \nu \Delta_x \eta_t + \Delta_x^2 \eta = -S_\eta(-pI + \nabla u)(x, \eta(x, t), t) n^\eta \cdot e_3 \qquad \text{on } \mathbb{T}^2 \times (0, T),$$

  where $\Delta_x = \partial_{11} + \partial_{22}$ and $S_\eta(x, t) = \sqrt{1 + |\nabla_x \eta(x, t)|^2}$
- fluid velocity conditions:

$$\begin{aligned} u(x, 0, t) = 0 \qquad & \text{on } \mathbb{T}^2 \times (0, T), \\ u(x, \eta(x, t), t) = (0, 0, \eta_t(x, t)) \qquad & \text{on } \mathbb{T}^2 \times (0, T). \end{aligned}$$

- initial conditions

$$u(x, z, 0) = u_0(x, z), \quad \eta(x, z, 0) = \eta_0(x, z), \quad \partial_t \eta(x, z, 0) = \eta_1(x, z).$$

Basic properties

- volume conservation

$$\frac{d}{dt}\int_{\mathbb{T}^2} \eta(x,t)dx = \ldots = \int_{\Omega_\eta(t)} \operatorname{div} u(x,z)dxdz = 0$$

- energy dissipation inequality

$$E(t) + \int_0^t D(s)ds \leq E(0)\,, \quad 0 \leq t < T,$$

where

$$E(t) = \frac{1}{2}\left(\|u(t)\|^2_{L^2(\Omega_\eta(t))} + \|\partial_t\eta(t)\|^2_{L^2(\mathbb{T}^2)} + \|\Delta_x\eta(t)\|^2_{L^2(\mathbb{T}^2)}\right),$$
$$D(t) = \|\nabla u(t)\|^2_{L^2(\Omega_\eta(t))} + \nu\|\partial_t\nabla_x\eta(t)\|^2_{L^2(\mathbb{T}^2)}$$

- in particular

$$\frac{1}{2}\|\Delta_x\eta(t)\|^2_{L^2(\mathbb{T}^2)} \leq E(0)\,, \quad 0 \leq t < T$$

First no-contact result

Condition on initial data

- using $H^2(\mathbb{T}^2) \hookrightarrow L^\infty(\mathbb{T}^2)$, there exists $C_S > 0$ such that

$$\|\eta(t) - \bar{\eta}(t)\|_{L^\infty(\mathbb{T}^2)} \leq C_S \|\Delta_x \eta(t)\|_{L^2(\mathbb{T}^2)} , \qquad 0 \leq t < T,$$

  where $\bar{\eta}(t) = \fint_{\mathbb{T}^2} \eta(t) dx$.
- since $\bar{\eta}(t) = \fint_{\mathbb{T}^2} \eta_0(x) dx = \bar{\eta}_0$,

$$\|\eta(t) - \bar{\eta}_0\|_{L^\infty(\mathbb{T}^2)} \leq C_S \|\Delta_x \eta(t)\|_{L^2(\mathbb{T}^2)} \leq C_S \sqrt{2E(0)} , \qquad 0 \leq t < T.$$

- therefore

$$\eta(x,t) \geq \bar{\eta}_0 - C_S \sqrt{2E(0)} , \qquad x \in \mathbb{T}^2, \quad 0 \leq t < T.$$

- choosing initial data (u_0, η_0, η_1) satisfying

$$E(0) \leq \frac{\bar{\eta}_0^2 (1-\kappa)^2}{2C_S^2} , \quad \text{for some } \kappa \in (0,1) ,$$

  it follows [BKLM-1]:

$$\eta(x,t) \geq \kappa \bar{\eta}_0 > 0 , \qquad x \in \mathbb{T}^2, \quad 0 \leq t < T.$$

First no-contact result

Application to micro-fluidics

- assumptions on initial data:

$$\eta_0(x) = \varepsilon h_0(x)\,, \quad \varepsilon \ll 1\,, \quad \int_{\mathbb{T}^2} h_0(x)dx = 1\,,$$

$$u_0(x,z) = z(z-\eta_0)\tilde{u}_0(x)\,, \quad (x,z) \in \mathbb{T}^2 \times (0,\eta_0(x))$$

$$\eta_1(x) = u_0(x,\eta_0(x)) \cdot e_3$$

- (for engineers) the condition becomes

$$O(\varepsilon^3) + \|\Delta_x h_0\|^2_{L^2(\mathbb{T}^2)} \le 256(1-\kappa)^2\,, \quad \text{for some } \kappa \in (0,1)$$

- an example: $h_0(x) = 1 + \frac{1-\kappa}{4}\left(\cos(2\pi x_1) + \cos(2\pi x_2)\right)$

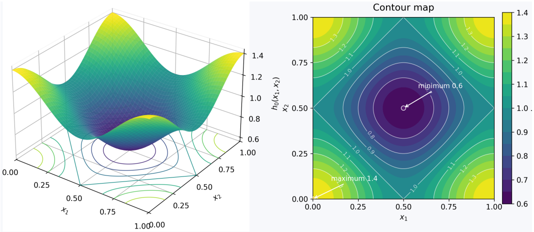


Figure: $h_0(x)$ with $\kappa = 0.2$

Second no-contact result

Inspired by

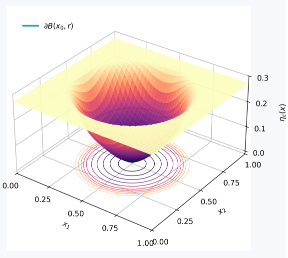
- C. Grandmont and M. Hillairet. *Existence of Global Strong Solutions to a Beam–Fluid Interaction System*. Arch. Rational Mech. Anal. 220 (2016), 1283-1333.
  - distance estimate: There exists a constant C_0 depending only on initial data such that
    $$\sup_{t\in(0,T)} \|\eta^{-1}(t,\cdot)\|_{L^1(\mathbb{T})} \leq C_0(1+T)\,.$$
  - an interpolation lemma: There exists a continuous function $D_{\min} : (0,\infty)\times[0,\infty)\to(0,\infty)$ such that, given a positive function $\eta \in H^2(\mathbb{T})$, there holds
    $$\|\eta^{-1}\|_{L^\infty(\mathbb{T})} \leq D_{\min}(\|\eta^{-1}\|_{L^1(\mathbb{T})}, \|\eta\|_{H^2(\mathbb{T})})\,.$$

- D. Breit and A. Roy, *Compressible fluids and elastic plates in 2D: A conditional no-contact theorem*, Physica D: Nonlinear Phenomena (2025): 134967.
  - under certain regularity assumptions on weak solutions there is no contact on $(0,T)$.
  - an explicit question: *Is it possible to obtain a conditional no-contact result in the 3D case?*

Second no-contact result

Interpolation lemma

- a 2D counterexample to 1D interpolation lemma: $\eta_c(x) = |x - x_0|^{3/2}$, $x \in B(x_0, r)$



Lemma

Let $\eta : \mathbb{T}^2 \to \mathbb{R}$ *be such that* $\inf_{\mathbb{T}^2} \eta > 0$, $\eta \in W^{2,\infty}(\mathbb{T}^2)$ *and* $\eta^{-1} \in L^1(\mathbb{T}^2)$. *Then* $\eta^{-1} \in L^\infty(\mathbb{T}^2)$, *and we have the estimate*

$$\|\eta^{-1}\|_{L^\infty(\mathbb{T}^2)} \leq \frac{8}{M}\left(\exp\left(\frac{LM}{2\pi}\right) - 1\right),$$

where $L = \|\eta^{-1}\|_{L^1(\mathbb{T}^2)}$ *and* $M = \|\eta\|_{W^{2,\infty}(\mathbb{T}^2)}$.

Second no-contact result

Regularity assumptions

- let (u,η) be a weak solution to the FPI system with initial data $(\eta_0,\eta_1,u_0)\in(H^2(\mathbb{T}^2),L^2(\mathbb{T}^2),L^2(\Omega_{\eta_0}))$ such that $\inf_{\mathbb{T}^2\times[0,T)}\eta>0$ and

$$u\in H^1((0,T),L^2(\Omega_\eta)^3)\,,$$
$$\eta\in L^\infty((0,T),H^{10/3}(\mathbb{T}^2))\cap H^1((0,T),H^1(\mathbb{T}^2))\,.$$

Proposition (Distance estimate)

There exists a constant $C>0$ *depending on the regularity of* (u,η) *and* (η_0,η_1,u_0) *such that*

$$\sup_{t\in(0,T)}\|\eta^{-1}(t,\cdot)\|_{L^1(\mathbb{T}^2)}\leq C\,.$$

Theorem (BKLM-2)

$$\sup_{t\in(0,T)}\|\eta^{-1}(t,\cdot)\|_{L^\infty(\mathbb{T}^2)}\leq C.$$

In particular, there is no contact on $[0,T]$*.*

Second no-contact result

Distance estimate – brief idea of the proof

- define $q_0(x,t) = -\dfrac{6}{\eta^2(x,t)}$ and

$$w_\alpha(x,z,t) = \frac{1}{2} z(z-\eta(x,t))\partial_\alpha q_0\,, \qquad \alpha = 1,2,$$
$$w_3(x,z,t) = \frac{1}{12} z^2(3\eta - 2z)\Delta_x q_0 + \frac{1}{4} z^2 \nabla_x \eta \cdot \nabla_x q_0$$

- $\operatorname{div} w = 0$, $w(x,\eta(x,t),t) = \Delta_x \eta e_3$ for $x \in \mathbb{T}^2,\ t \in (0,T)$, and q_0 satisfies

$$\partial_\alpha q_0 = \partial_3^2 w_\alpha\,, \quad \alpha = 1,2$$

- use $(\Delta_x \eta, w)$ as test functions to get

$$6\int_{\mathbb{T}^2} \frac{1}{\eta}\Big|_0^t + \int_0^t \int_{\mathbb{T}^2} |\nabla_x \Delta_x \eta|^2 + \int_{\mathbb{T}^2} \frac{\nu}{2} |\Delta_x \eta|^2 \Big|_0^t$$
$$= \int_0^t \int_{\mathbb{T}^2} |\nabla_x \eta_t|^2 + \int_{\mathbb{T}^2} \eta_t \Delta_x \eta \Big|_0^t + \int_0^t \int_{\Omega_\eta} A(w) : \nabla u + \int_0^t \int_{\Omega_\eta} w u_t$$
$$+ \int_0^t \int_{\mathbb{T}^2} (u \cdot \nabla u) w$$

Second no-contact result

Distance estimate – brief idea of the proof

- above

$$A(w) = \begin{pmatrix} \partial_1 w_1 & \partial_2 w_1 & -\partial_1 w_3 \\ \partial_1 w_2 & \partial_2 w_2 & -\partial_2 w_3 \\ \partial_1 w_3 & \partial_2 w_3 & 0 \end{pmatrix}.$$

- estimating all terms:

$$\int_{\mathbb{T}^2} \frac{1}{\eta(t)} \leq C + C \int_0^t \int_{\mathbb{T}^2} \frac{1}{\eta(s)}, \quad t \in (0, T),$$

where C depends on the assumed regularity of (u, η) and (u_0, η_0, η_1)

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