# The periodic Kelvin problem

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Humboldt-Universität zu Berlin

Based on joint works with A. Cesaroni, F. Nobili & E. Paolini

**How little interface is needed to partition space periodically into prescribed volume cells?**

volume constraints $\Longrightarrow$ free boundaries $\Longrightarrow$ optimal periodic patterns

**2D**
Honeycombs, double tilings, explicit profiles

**3D**
Parallelohedra, truncated octahedron

**4D**
Parallelohedra, 24-cell?

# Periodic tilings

A **periodic** $N$**-tiling** is a collection of measurable sets $\mathbf{T}$ in $\mathbb{R}^d$ given by

$$\mathbf{T} = \{(E_i + g) \colon g \in G, i = 1, \dots, N\},$$

where $G$ is a **lattice** (a discrete group of translations) and $(E_i)_{i=1}^N$ satisfies

$$|(E_i + g) \cap (E_j + h)| = 0, \quad \text{for } (i,g) \neq (j,h), \qquad |\mathbb{R}^d \setminus \cup_{i,g}(E_i + g)| = 0.$$

In this case, we call $\mathbf{T}$ an $N$-tiling relative to $G$.

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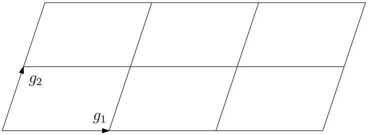
*Remarks.*

- There are $d$ linearly independent vectors $g_i \in G$ such that

$$\forall g \in G \quad \Longrightarrow \quad g = \sum_i \alpha_i g_i \text{ for some } \alpha_i \in \mathbb{Z}.$$

  We define $\mathrm{Area}(G)$ to be the area spanned by the parallelogram formed by the vectors $g_i$.




- $D := \cup_i E_i$ is a **fundamental domain** for $G$ and satisfies $|D| = \mathrm{Area}(G)$.

# Perimeter of tilings

The **perimeter** of a periodic $N$-tiling $\mathbf{T}$ is given by

$$\mathrm{Per}(\mathbf{T}) := \frac{1}{2}\sum_{i=1}^{N} \mathrm{Per}(E_i),$$

where $\mathrm{Per}(E)$ is the Caccioppoli perimeter of a measurable set $E$.

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## Definition

An $N$-tiling $\mathbf{T} = \{(E_i + g) \colon g \in G, i = 1, \ldots, N\}$ is **isoperimetric relative to** $G$ if

$$\mathrm{Per}(\mathbf{T}) \leq \mathrm{Per}(\mathbf{T}'), \qquad (*)$$

for all $N$-tilings $\mathbf{T}'$ relative to $G$ satisfying

$$\mathbf{T}' = \{(E_i' + g) \colon g \in G, i = 1, \ldots, N\} \quad \text{with} \quad |E_i| = |E_i'|, \quad \forall i = 1, \ldots, N.$$

Furthermore, a periodic $N$-tiling $\mathbf{T}$ is **isoperimetric** if $(*)$ holds for all $N$-tilings $\mathbf{T}'$ as above relative to any lattice $G'$ with $\mathrm{Area}(G') = \sum_i |E_i|$.

# The Isoperimetric problem

Consider $N \in \mathbb{N}$, a lattice $G$ in $\mathbb{R}^d$, and $(v_1, \ldots, v_N) \in \mathbb{R}^N_+$, $\Lambda = \sum_i v_i = \mathrm{Area}(G)$.

## Questions

(?) (Existence) Are there isoperimetric $N$-tiling $\mathbf{T}$ with $|E_i| = v_i$?

(?) (Regularity) What can we say about isoperimetric tilings?

(?) (Classification) At least for $d = 2$ and $N = 1, 2$, can we classify them?

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*Related literature*

- Honeycomb conjecture [Hales '99]
- Isoperimetric fundamental domains on $3$-dimensional manifolds [Choe '89]
- Minimal spines on Riemannian surfaces and tori [MartelliNovagaPludaRiolo '17]
- Isoperimetric fundamental domains for general perimeters [CesaroniFragalàNovaga '25]
- Bipartite tilings with minimal interface [FortesTexeira '01],[FortesGranerTexeira '02]

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*Further literature*

- ▷ Cluster theory (Morgan's book and Milman's ICM proceedings).

Main results: existence, regularity, and classification

# Existence

## Theorem (NobiliNovaga '24)

Let $N \in \mathbb{N}$, let $G$ be a lattice in $\mathbb{R}^d$ and let $(v_1, \ldots, v_N) \in \mathbb{R}^N_+$, $\Lambda = \sum_i v_i = \mathrm{Area}(G)$. There exists a minimiser of

$$\inf \left\{ \mathrm{Per}(\mathbf{T}) \colon \mathbf{T} = \{(E_i + g) \colon i = 1, \ldots, N,\ g \in G\},\ |E_i| = v_i \right\}.$$

In particular, a minimizer is an isoperimetric $N$-tiling relative to $G$. Moreover, there exists a minimizer of

$$\inf_{G, \mathrm{Area}(G) = \Lambda} \inf \left\{ \mathrm{Per}(\mathbf{T}) \colon \mathbf{T} = \{(E_i + g) \colon i = 1, \ldots, N,\ g \in G\},\ |E_i| = v_i \right\}.$$

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*Strategy* (direct method)

- Minimizing sequence $\mathrm{Per}(\mathbf{T}_n) \to \inf$ as $n \to \infty$
- Concentration compactness by $G$-translations
- $G$-invariance and lower semicontinuity of the perimeter

# Regularity in the planar case

**Theorem (NobiliNovaga '24)**

Let $N \in \mathbb{N}$, let $G$ be a lattice in $\mathbb{R}^2$. Suppose that $\mathbf{T} := \{(E_i + g) : i = 1, \ldots, N,\ g \in G\}$ is isoperimetric relative to $G$. Then, for every $i = 1, \ldots, N$ we have

i) $E_i$ has an **open and bounded** representative (abusing notation from here on)

ii) $E_i$ is **regular**: there are finitely many connected components and $\partial(E_i + g)$ is made up of finitely many either straight line segments or circular arcs (edges)

iii) all the edges meet at **triple points** with equal angles of $120^\circ$

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*Remarks.*

▷ Obstacle. For every ball $B \subset \mathbb{R}^d$, it is possible that

$$\#\{(i, g) \colon |(E_i + g) \cap B| > 0\} = +\infty.$$

In other words, the partition $(E_i + g)_{i,g}$ is not a priori locally finite

▷ Key point. In dimension $d = 2$ the perimeter controls the diameter of connected components.

▷ For $d > 2$ and $N = 1$ regularity holds [CesaroniNovaga '24].

# Double tilings

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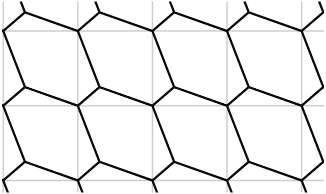
*Remarks* (hexagons play a role).

- If $|E_1| = 0$ or $|E_2| = 0 \rightsquigarrow$ degenerate $\mathbf{T} = \{E + g \colon g \in G\}$ with $|E| = \mathrm{Area}(G)$
  $\rightsquigarrow E$ is a hexagon compatible with $G$
- If $|E_1| = |E_2| = \frac{1}{2}\mathrm{Area}(G)$, a good guess is that $E_1, E_2$ are adjacent hexagons



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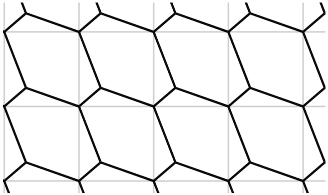


- Intermediate configurations are the main challenges

# Classification

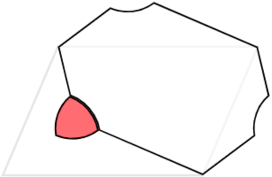
**Theorem (NobiliNovagaPaolini, '25)**

Let $\mathbf{T} = \{(E_i + g) \colon i \in \{1,2\}, g \in G\}$ be a non-degenerate isoperimetric double tiling relative to a lattice $G$. Then, $E_1, E_2$ are curvilinear polygons with angles of $120^\circ$ and, up to exchanging $E_1, E_2$, one of the following holds:

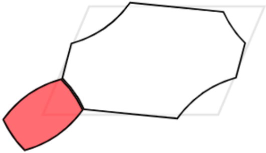
(3,9) $E_1$ is a Reuleaux triangle and $E_2$ is a chipped hexagon

(4,8) $E_1$ is a strictly convex curvilinear quadrilateral and $E_2$ is a chipped octagon

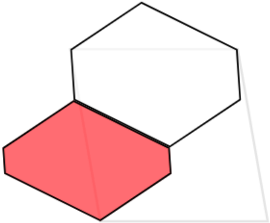
(6,6) $E_1, E_2$ are hexagons with flat edges



Conf. (3,9)

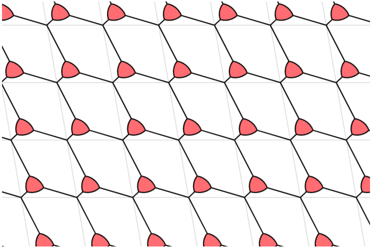


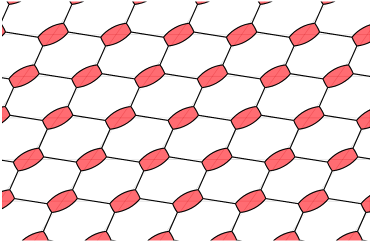
Conf. (4,8)

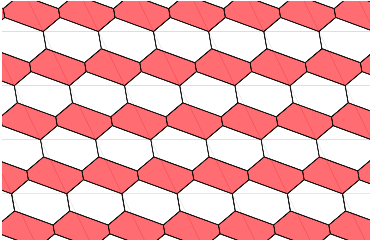


Conf. (6,6)

# Classification I







# Classification II

*Remarks.*

- For a given lattice $G$, we do not know whether all configurations are isoperimetric
- If $\mathbf{T}$ is isoperimetric relative to $G$, we do not know the explicit value of $\mathrm{Per}(\mathbf{T})$

# Classification II

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- ▷ For a given lattice $G$, we do not know whether all configurations are isoperimetric
- ▷ If $\mathbf{T}$ is isoperimetric relative to $G$, we do not know the explicit value of $\mathrm{Per}(\mathbf{T})$

**Goal**. Address the above for isoperimetric double tilings (i.e. allowing $G$ to vary)

*Why*?

- ▷ Minimizing over all possible lattices selects more symmetric configurations
- ▷ Computations become accessible
- ▷ Derive explicitly the energy profile and study the phase transition points

# Isoperimetric profile

## Definition

Let $G$ be a lattice in $\mathbb{R}^2$ with $\mathrm{Area}(G) = 1$. We define the **isoperimetric profile function relative to** $G$ by

$$[0,1] \ni x \mapsto \mathcal{I}_G(x) := \inf \left\{ \mathrm{Per}(\mathbf{T}) \colon \mathbf{T} = \{(E_i + g) \colon i \in \{1,2\}, g \in G\}, \begin{array}{l} |E_1| = x \\ |E_2| = 1 - x \end{array} \right\}$$

Furthermore, we define the **isoperimetric profile function**

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*Remark*. The previous remarks can be rephrased using profile functions

**Goal**.

- ▷ Compute $x \mapsto \mathcal{I}(x)$
- ▷ Give an explicit description of minimizing configurations $\mathrm{Per}(\mathbf{T}(x)) = \mathcal{I}(x)$
- ▷ Discuss the lattice $G(x)$, uniqueness of $\mathbf{T}(x)$ and the phase transitions

## Theorem (NobiliNovagaPaolini, '25)

There are explicit points $x_1, x_2 \in \left(0, \frac{1}{2}\right)$ (approx. $x_1 \approx 0.062$, $x_2 \approx 0.317$) such that:

$$\mathcal{I}(x) = \begin{cases} \sqrt{2}\sqrt{\pi - \sqrt{3}}\sqrt{x} + \sqrt[4]{12} & \text{if } x \in [0, x_1), \\ 2\sqrt{\frac{\pi}{3} + 1 - \sqrt{3}}\sqrt{x} + 2 & \text{if } x \in [x_1, x_2), \\ 2\sqrt[4]{3} & \text{if } x \in (x_2, \frac{1}{2}], \\ \mathcal{I}(1-x) & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

In particular, $x \mapsto \left(\mathcal{I}(x) - \mathcal{I}(0)\right)^2$ is concave.

# Isoperimetric profile I

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In particular, $x \mapsto (\mathcal{I}(x) - \mathcal{I}(0))^2$ is concave. Moreover, if $\mathrm{Per}(\mathbf{T}(x)) = \mathcal{I}(x)$, then:

- $(o)$ $\mathbf{T}(0)$ is the Honeycomb tiling;
- $(i)$ if $0 < x < x_1$, then $\mathbf{T}(x)$ is a $(3, 9)$ configuration;
- $(ii)$ if $x = x_1$, then $\mathbf{T}(x)$ can either be a $(3, 9)$ or a $(4, 8)$ configuration;
- $(iii)$ if $x_1 < x < x_2$, then $\mathbf{T}(x)$ is a $(4, 8)$ configuration;
- $(iv)$ if $x = x_2$, then $\mathbf{T}(x)$ can either be a $(4, 8)$ or a $(6, 6)$ configuration;
- $(v)$ if $x_2 < x < \frac{1}{2}$, then $\mathbf{T}(x)$ is a $(6, 6)$ configuration.

Finally, if $x \notin \{x_1, x_2, 1 - x_2, 1 - x_1\}$, then a minimizer $\mathbf{T}(x)$ is unique up to isometries.

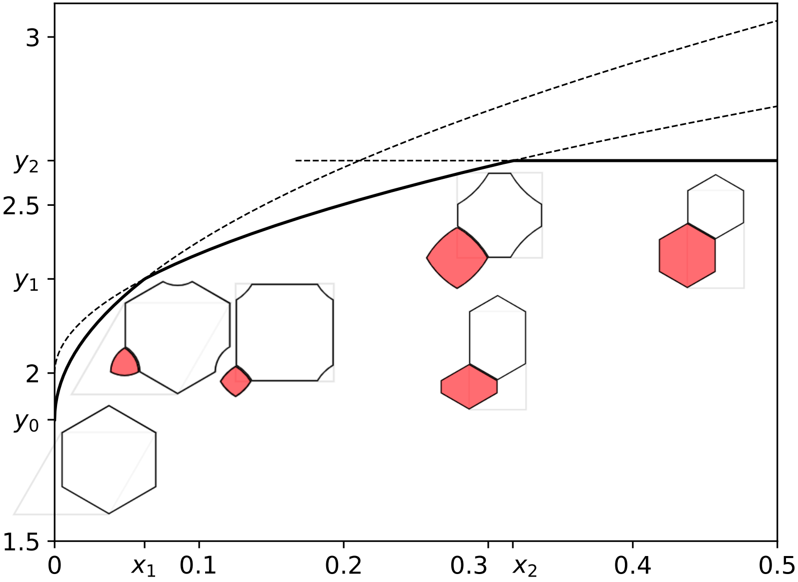
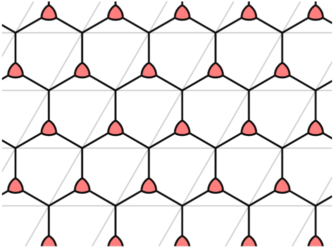
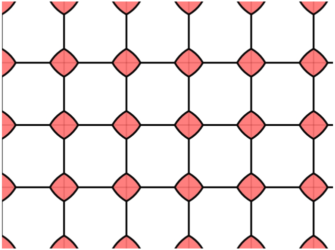



Figure: Plot of $\mathcal{I}(x)$ against $x \in [0, \frac{1}{2}]$. We set $y_i := \mathcal{I}(x_i)$ for $i = 1, 2$.

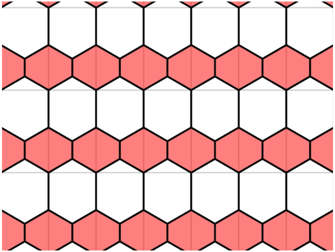
# Minimizing lattice



$G(x)$ is the Honeycomb lattice



$G(x)$ is the square lattice



$G(x)$ is a rectangular lattice

# Extensions and open problems

**Isoperimetric $N$-tilings**

(?) For which $N \in \mathbb{N}$ and $\vec{v} := (v_1, \dots, v_N)$ can we classify isoperimetric $N$-tilings?

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*(Qualitative) partial answer*. Almost equal area regime.

**Theorem (NobiliNovaga, '24)**

For every $N \in \mathbb{N}$, there exists $\delta := \delta(N) \in (0, 1/2)$ such that the following holds. For every $\vec{v} := (v_1, \dots, v_N) \in \mathbb{R}^N_+$ such that

$$\max_{i=1,\dots,N} |v_i - 1| < \delta, \qquad \sum_{i=1}^{N} v_i = N,$$

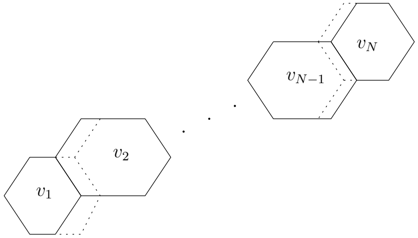
we have

$$\frac{N}{2}\mathrm{Per}(H) = \inf_{G,\mathrm{Area}(G)=N} \inf \{\mathrm{Per}(\mathbf{T})\colon \mathbf{T} = \{(E_i + g)\colon |E_i| = v_i, i \in \{1, \dots, N\}, g \in G\}$$

where $H \subset \mathbb{R}^2$ is a regular hexagon with $|H| = 1$.

# Comments

Given $\vec{v} = (v_1, \dots, v_N)$ as in the assumptions, consider the sets $(E_i)$:



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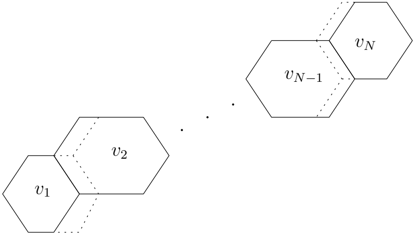


- The periodic $N$-tiling $\mathbf{T}$ generated by $(E_i)$ satisfies $\mathrm{Per}(\mathbf{T}) = \frac{N}{2}\mathrm{Per}(H)$. Hence, it is a minimizing competitor for the problem.

- The result gives a qualitative stability of the Honeycomb partition among tilings enclosing $N$ almost equal areas

# Open problems

**Fixed lattice**. Let $G$ be a lattice with $\mathrm{Area}(G) = 1$

(?) For which $x \in [0, 1]$ is the configuration $(4, 8)$ admissible?

(?) For which $x \in [0, 1]$ is the configuration $(4, 8)$ also isoperimetric?

(?) What is the value of $x \mapsto \mathcal{I}_G(x)$? At which point(s) do phase transitions occur?

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**Non-periodic scenario** (Kelvin problem with two cells)

(?) For $x \in [0, 1]$, is the partition of $\mathbb{R}^2$ induced by the isoperimetric double tiling $\mathbf{T}(x)$ a *local perimeter minimizer?*

*Partial answer*. We have evidence for the existence of two disjoint subintervals $I, J \subset [0, \frac{1}{2}]$ where $\mathbf{T}(x)$ is cannot be a local perimeter minimizer.

# Anisotropic perimeter

In the plane $\mathbb{R}^2$, consider the *Manhattan norm* and its dual norm

$$\|(x,y)\|_{\ell_1} := |x| + |y|.$$

Given a lattice $G$ and a $N$-tiling $\mathbf{T}$ relative to $G$, we consider the $\ell_1$-anisotropic perimeter

$$\mathrm{Per}_{\ell_1}(\mathbf{T}) := \frac{1}{2}\sum_{i=1}^{N} \mathrm{Per}_{\ell_1}(E_i),$$

where, for every set of finite perimeter $E \subset \mathbb{R}^2$, we define

$$\mathrm{Per}_{\ell_1}(E) := \int_{\partial^* E} \|\nu_E(x)\|_{\ell_1}\, \mathrm{d}\mathcal{H}^1(x).$$

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$$\mathrm{Per}_{\ell_1}(E) := \int_{\partial^* E} \|\nu_E(x)\|_{\ell_1}\, \mathrm{d}\mathcal{H}^1(x).$$

**Wulff isoperimetric inequality.** It holds that

$$\mathrm{Per}_{\ell_1}(E) \geq 4\sqrt{|E|}, \qquad \forall E \subset \mathbb{R}^2,$$

with equality if and only if $E$ is equivalent to a square of area $|E|$ with edges parallel to the coordinate axes.

# $\ell_1$-Isoperimetric profiles

As before, we consider for all $x \in [0,1]$ the $\ell_1$-isoperimetric profile functions

$$\mathcal{I}_{G,\ell_1}(x) := \inf \left\{ \mathrm{Per}_{\ell_1}(\mathbf{T}) \colon \mathbf{T} = \{(E_i + g) \colon i \in \{1,2\}, g \in G\}, \begin{array}{l} |E_1| = x \\ |E_2| = 1 - x \end{array} \right\},$$

$$\mathcal{I}_{\ell_1}(x) := \inf\{\mathcal{I}_{G,\ell_1}(x) \colon G \text{ lattice with } \mathrm{Area}(G) = 1\}.$$

**Theorem**

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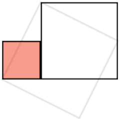
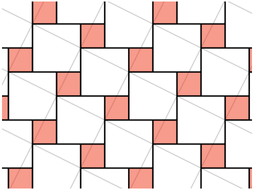
But the *Pythagorean tiling* is a competitor $\mathbf{T}(x)$ giving the upper bound. $\square$





# $\ell_1$-Isoperimetric double tilings with rectangular periodicity

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- In the $\ell_1$-anisotropic setting, fixing a lattice $G$ seems necessary
- Rectangular lattices are already interesting enough.

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**Theorem (NobiliNovagaPaolini, '26+)**

*Let $G$ be the rectangular lattice generated by $(a,0),(0,b)$ with $\mathrm{Area}(G) = ab = 1, a \geq b$. Then, it holds*

$$\mathcal{I}_{G,\ell_1}(x) = \begin{cases} 2\sqrt{x} + a + b & \textit{if } x \leq \frac{b^2}{4}, \\ a + 2b & \textit{if } \frac{b^2}{4} < x \leq \frac{1}{2}, \\ \mathcal{I}_{G,\ell_1}(1-x) & \textit{otherwise.} \end{cases}$$

*In particular, $x \mapsto (\mathcal{I}_{G,\ell_1}(x) - \mathcal{I}_{G,\ell_1}(0))^2$ is concave.*

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*In particular, $x\mapsto(\mathcal{I}_{G,\ell_1}(x)-\mathcal{I}_{G,\ell_1}(0))^2$ is concave. Finally, we have:*

i) *if $x=0$, then $\mathbf{T}(0)$ generated by $D=[0,a]\times[0,b]$ is minimizing;*

ii) *if $0<x\leq\frac{b^2}{4}$, then $\mathbf{T}(x)$ generated by a square and a chipped rectangle is minimizing;*

iii) *if $\frac{b^2}{4}\leq x\leq\frac{1}{2}$, then $\mathbf{T}(x)$ generated by two adjacent rectangles is minimizing.*

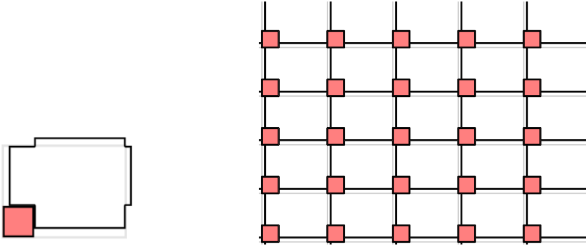
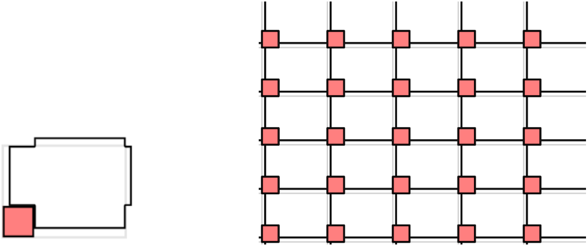


Figure: A $\ell_1$-isoperimetric double tiling $\mathbf{T}(x)$ when $0 < x \leq \frac{b^2}{4}$.



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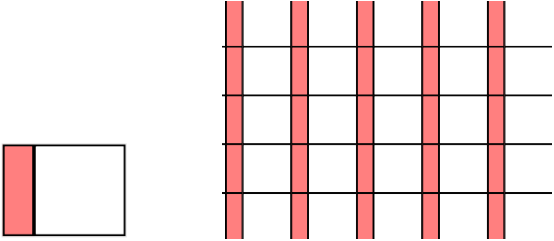

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# Uniqueness

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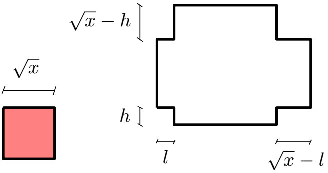
- the $\ell_1$-anisotropy is invariant under translations, but not under arbitrary rotations
- the following modifications are invariant for the $\ell_1$-perimeter



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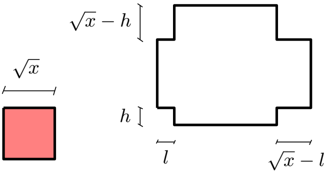
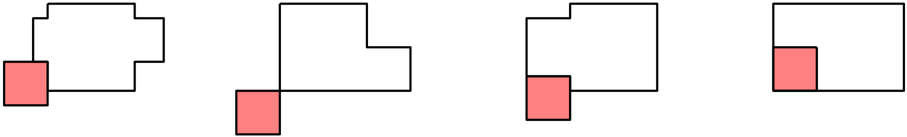



*Examples.* Choosing different parameters $l, h \in [0, \sqrt{x}]$, we have



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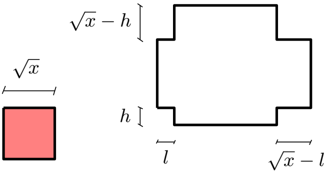


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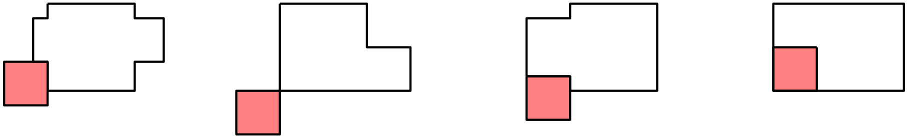

*Target* (in progress). Uniqueness up to translations and up to the above choices

# Higher dimensions, $N = 1$

The periodic Kelvin problem, as well as the full Kelvin problem, is open in any dimension $d > 2$, there are however some natural conjectures:

- (Kelvin conjecture) for $d = 3$ the minimizer is expected to be the so-called **Kelvin cell**, with lattice BCC, which is obtained from the regular truncated octahedron;
- ($24$-cell conjecture) for $d = 4$ the minimizer should be the $24$-cell polytope, which is the Voronoi cell of the lattice $D_4$.

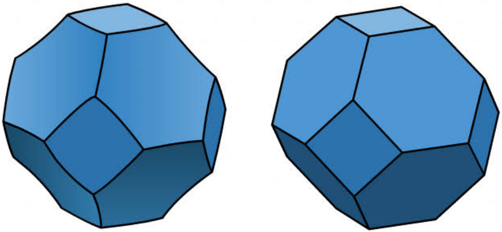

Figure: The Kelvin cell and the truncated octahedron

# Parallelohedra: a constrained periodic Kelvin problem

A **parallelohedron** is a convex polyhedron whose translates tile $\mathbb{R}^d$ face-to-face.

- ▷ Fedorov: exactly five combinatorial types in dimension three.
- ▷ The regular truncated octahedron is the Voronoi cell of the body-centred cubic lattice (BCC).
- ▷ The regular rhombic dodecahedron is the Voronoi cell of the face-centred cubic lattice (FCC).

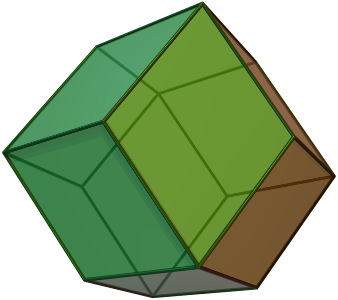

Figure: The regular rombic dodecahedron

# Parallelohedra: a constrained periodic Kelvin problem

**Question:** which parallelohedron in $\mathbb{R}^3$, of fixed volume, has the least surface area?

**Local minimality [CesaroniNovaga '26a, Song '26]**

- ▷ The truncated octahedron is a **strict local minimizer** of the isoperimetric quotient;
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### The truncated octahedron conjecture [Bezdek '06]

The truncated octahedron is the **global minimizer** of the isoperimetric quotient among three-dimensional parallelohedra.

# The truncated octahedron conjecture

**Theorem (Hales-Song '26, Cesaroni-Novaga '26b)**

For every three-dimensional parallelohedron $P$,

$$\mathrm{Iso}(P) := \frac{\mathcal{H}^2(\partial P)}{|P|^{2/3}} \geq \frac{3(1+2\sqrt{3})}{4^{2/3}},$$

with equality if and only if $P$ is similar to the regular truncated octahedron.

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If $P$ is not of truncated-octahedral Fedorov type, i.e. it has at most $12$ facets, then the stronger bound holds

$$\mathrm{Iso}(P) \geq 3\, 2^{5/6},$$

with equality if and only if $P$ is similar to the regular rombic dodecahedron.

**This settles the periodic Kelvin's problem in $\mathbb{R}^3$, under convexity assumption.**

# Proof sketch

- Every 3D lattice admits an **obtuse superbase**; its geometry is encoded by six non-negative Selling parameters. BCC corresponds to $(1,1,1,1,1,1)$, FCC to $(0,1,1,1,1,0)$.
- The Voronoi cell is a **zonotope**; volume and surface area admit explicit formulae in terms of the six parameters [CesaroniNovaga '26a].
- Algebraic estimates yield the BCC minimality among Voronoi cells.
- A determinant inequality controls affine deformations, extending the result from Voronoi cells to arbitrary parallelohedra.

# Open problems

- **(The four-dimensional case)** Is the regular $24$-cell, that is the Voronoi cell of $D_4$, the unique minimizer up to similarity?
- **(Lattices of Voronoi's first kind)** For general $d$, restrict to lattices admitting an obtuse superbase. Is the Voronoi cell of $A_d^*$ (permutahedron) optimal?
- **(Asymptotic behavior)** For general $d$, let $\gamma_d := \min_P \mathrm{Iso}(P)$. Determine the behavior of $\gamma_d$ as $d \to +\infty$.
- **(Quantitative stability)** Find a global estimate of the form

$$\mathrm{Iso}(P) - \mathrm{Iso}_{BCC} \geq c\, \mathrm{dist}(P, BCC)^2.$$

- **(Beyond parallelohedra)** Find the minimum among equal-volume fundamental domains (periodic Kelvin problem).

# Thank you!

(Pictures)
https://paolini.github.io/double-tiling/